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VARIATION FORMULAS OF SOLUTION FOR A CLASS OF CONTROLLED DIFFERENTIAL EQUATION WITH DELAY IN THE PHASE COORDINATES AND CONTROLS

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Abstract. For nonlinear controlled functional differential equations variation formulas of solution are proved, in which the effects of perturbations of the initial moment and constant delays, and also that of the continuous initial condition are detected.

Keywords: *Delay controlled equation, variation formula of solution, effect of delay perturbation*.

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1. Introduction

As is known the real processes contain information about their behavior in the past and are described by delay differential equations (Hale, 1977; Kharatishvili & Tadumadze, 2000). To illustrate this, below we will consider a simplest model of the flying apparatus. Let a flying apparatus be controlled at the process of flight by radio signal $u(t)$. A radio signal, as a rule, is weak, because it is necessary its transformation and intensification. For this need a defined time τ. The number τ is the so-called delay and its quantity depends on transforming equipment, which is placed on the flying object. A simplified classical mathematical model corresponding to this situation is the following system of differential equations

$$\begin{cases} L(p)x = y(t-\tau), \\ M(p)y = u(t). \end{cases}$$

Here $L(p)$ and $M(p)$ are given linear differential operators with constant coefficients; first equation describes movement of the flying apparatus and $y(t)$ is transformed signal, which is worked out by the second system.

Linear representation of the main part of the increment of a solution with respect to perturbations of the initial data of a differential equation is called the variation formula of solution (variation formula). In this paper, the essential novelty is that here the variation formula is proved when simultaneously occurs perturbations of the initial moment and delays as well as in the phase coordinates and controls. The term "variation formula of solution" has been introduced by R.V. Gamkrelidze and proved in (Gamkrelidze, 1978) for the ordinary differential equation. The effects of perturbation of the initial moment and the discontinuous initial condition in the variation formulas for the first time were revealed in (Tadumadze, 2000) for the delay differential equation. The variation formula plays a basic role in proving the necessary conditions of optimality (Gamkrelidze, 1978; Kharatishvili & Tadumadze, 2007; Tadumadze, 2017) and in the sensitivity analysis of mathematical models (Marchuk, 1994). Moreover, the variation formula allows one to construct an approximate solution of the perturbed equation.

In the present work for the controlled delay differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau), u(t), u(t-\theta))$$

with the continuous initial condition the variation formulas are proved. The continuity of the initial condition means that the values of the initial function and the trajectory always coincide at the initial moment. The variation formulas for various classes of controlled delay differential equations without delay in controls are derived in (Tadumadze & Alkhazishvili, 2004; Tadumadze & Nachaoui, 2011).

2. Formulation of the main result

Let $I = [a,b]$ be a finite interval and let $0 < \tau_1 < \tau_2$, $0 < \theta_1 < \theta_2$ be given numbers; suppose that $O \subset R^n$ and $U \subset R^n$ are open sets. Let the n-dimensional function $f(t, x, x_1, u, u_1)$ be continuous and bounded on $I \times O^2 \times U^2$ and continuously differentiable with respect to x, x_1, u, u_1. Let Φ and Ω be sets of continuously differentiable functions $\varphi : I_1 = [\hat{\tau}, b] \to O$ and $u : I_2 = [\hat{\theta}, b] \to U$, respectively, where $\hat{\tau} = a - \tau_2$ and $\hat{\theta} = a - \theta_2$.

To each element $\mu = (t_0, \tau, \theta, \varphi, u) \in \Lambda = [a,b) \times [\tau_1, \tau_2] \times [\theta_1, \theta_2] \times \Phi \times \Omega$ we assign the delay controlled differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau), u(t), u(t-\theta)) \tag{1}$$

with the continuous initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0]. \tag{2}$$

Definition 1. Let $\mu = (t_0, \tau, \theta, \varphi, u) \in \Lambda$. A function $x(t) = x(t;\mu) \in O$, $t \in [\hat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element μ and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (2) and is continuously differentiable on the interval $[t_0, t_1]$ and satisfies equation (1) on $[t_0, t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, \theta_0, \varphi_0, u_0) \in \Lambda$ be a fixed element and let $x_0(t)$ be the solution corresponding to the element μ_0 and defined on the interval $[\hat{\tau}, t_{10}]$, where $t_{00}, t_{10} \in (a,b)$, $t_{00} < t_{10}$, $\tau_0 \in (\tau_1, \tau_2)$, $\theta_0 \in (\theta_1, \theta_2)$.

Let us introduce the set of variation:

$$V = \Big\{ \delta\mu = (\delta t_0, \delta\tau, \delta\theta, \delta\varphi, \delta u) : \delta t_0 \in (a,b) - t_{00},\ \delta\tau \in (\tau_1, \tau_2) - \tau_0,\ \delta\theta \in (\theta_1, \theta_2) - \theta_0,$$
$$\delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i,\ \delta u = \sum_{i=1}^{k} \lambda_i \delta u_i,\ |\delta t_0| \le \alpha, |\delta\tau| \le \alpha, |\delta\theta| \le \alpha, |\lambda_i| \le \alpha, i = \overline{1,k} \Big\},$$

where $(a,b) - t_{00} = \{\delta t_0 = t_0 - t_{00} : t_0 \in (a,b)\}$ and $\alpha > 0$ is a fixed number.

There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times V$ we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$ and a solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to it (see Tadumadze, 2017, p.17, Theorem 1.4).

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\hat{\tau}, t_{10} + \delta_1]$. Therefore in the sequel the solution $x_0(t)$ is assumed to be defined on the interval $[\hat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t; \mu_0)$

$$\Delta x(t) = \Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), (t, \varepsilon, \delta\mu) \in [\hat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times V. \quad (3)$$

Theorem 1. For each $\hat{t}_0 \in (t_{00}, t_{10})$ there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary $(t, \varepsilon, \delta\mu) \in [\hat{t}_0, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V$ we have

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu), \quad (4)$$

where $\delta x(t; \delta\mu)$ on $[\hat{t}_0, t_{10} + \delta_2]$ has the form

$$\delta x(t; \delta\mu) = Y(t_{00}; t)\left\{\delta\varphi(t_{00}) + \left[\dot{\varphi}_0(t_{00}) - f[t_{00}]\right]\delta t_0\right\} - \left[\int_{t_{00}}^{t} Y(\xi; t) f_{x_1}[\xi]\dot{x}_0(\xi - \tau_0)d\xi\right]\delta\tau -$$

$$- \left[\int_{t_{00}}^{t} Y(\xi; t) f_{u_1}[\xi]\dot{u}_0(\xi - \theta_0)d\xi\right]\delta\theta + \int_{t_{00} - \tau_0}^{t_{00}} Y(\xi + \tau_0; t) f_{x_1}[\xi + \tau_0]\delta\varphi(\xi)d\xi +$$

$$+ \int_{t_{00}}^{t} Y(\xi; t)\left[f_u[\xi]\delta u(\xi) + f_{u_1}[\xi]\delta u(\xi - \theta_0)\right]d\xi; \quad (5)$$

furthermore, $Y(\xi; t)$ is the $n \times n$-matrix function satisfying the equation

$$Y_\xi(\xi; t) = -Y(\xi; t) \; f_x[\xi] - Y(\xi + \tau_0; t) f_{x_1}[\xi + \tau_0], \; \xi \in [t_{00}, t] \quad (6)$$

and the condition

$$Y(\xi; t) = \begin{cases} \mathrm{H}, \; for \; \xi = t, \\ \Theta, \; for \; \xi > t. \end{cases} \quad (7)$$

Here $f[t] = f(t, x_0(t), x_0(t - \tau_0), u_0(t), u_0(t - \theta_0)), H$ is the identity matrix and Θ is the zero matrix;

$\lim\limits_{\varepsilon \to 0} \dfrac{o(t; \varepsilon\delta\mu)}{\varepsilon} = 0$ uniformly for $(t, \delta\mu) \in [t_{00}, t_{10} + \delta_2] \times V^-$.

Some comments. The function $\delta x(t; \delta\mu)$ is called the first variation of the solution $x_0(t), t \in [\hat{t}_0, t_{10} + \delta_2]$ and the expression (5) is called the variation formula.

The expression $Y(t_{00}; t)\left[\dot{\varphi}_0(t_{00}) - f[t_{00}]\right]\delta t_0$ in formula (5) is the effect of the continuous initial condition (2) and perturbation of the initial moment t_{00}.

The addend

$$- \left[\int_{t_{00}}^{t} Y(\xi; t) f_{x_1}[\xi]\dot{x}_0(\xi - \tau_0)d\xi\right]\delta\tau$$

in formula (5) is the effect of perturbation of the delay τ_0.

The expression

$$-\left[\int_{t_{00}}^{t} Y(\xi;t) f_{u_1}[\xi]\dot{u}_0(\xi-\theta_0)d\xi\right]\delta\theta$$

in formula (5) is the effect of perturbation of the delay θ_0.

The expression

$$Y(t_{00};t)\delta\varphi(t_{00})+\int_{t_{00}-\tau_0}^{t} Y(\xi+\tau_0;t) f_{x_1}[\xi+\tau_0]\delta\varphi(\xi)d\xi$$

in formula (5) is the effect of perturbation of the initial function $\varphi_0(t)$.

The expression

$$\int_{t_{00}}^{t} Y(\xi;t)\big[f_u[\xi]\delta u(\xi)+f_{u_1}[\xi]\delta u(\xi-\theta_0)\big]d\xi$$

in formula (5) is the effect of perturbation of the controlled function $u_0(t)$.

For the linear equation

$$\dot{x}(t)=A(t)x(t)+B(t)x(t-\tau)+C(t)u(t)+D(t)u(t-\theta)$$

formula (5) has the form

$$\delta x(t;\delta\mu)=Y(t_{00};t)\{\delta\varphi(t_{00})+[\dot{\varphi}_0(t_{00})-A(t_{00})x_0(t_{00})-B(t_{00})\varphi_0(t_{00}-\tau_0)-$$

$$-C(t_{00})u_0(t_{00})-D(t_{00})u_0(t_{00}-\theta_0)]\delta t_0\}-\left[\int_{t_{00}}^{t} Y(\xi;t)B(\xi)\dot{x}_0(\xi-\tau_0)d\xi\right]\delta\tau-$$

$$-\left[\int_{t_{00}}^{t} Y(\xi;t)D(\xi)\dot{u}_0(\xi-\theta_0)d\xi\right]\delta\theta+\int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)B(\xi+\tau_0)\delta\varphi(\xi)d\xi+$$

$$+\int_{t_{00}}^{t} Y(\xi;t)[C(\xi)\delta u(\xi)+D(\xi)\delta u(\xi-\theta_0)]d\xi.$$

3. Auxiliary assertions

Lemma 1. There exist numbers $\varepsilon\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that

$$\max_{t\in[\hat{\tau},\, t_{10}+\delta_2]}|\Delta x(t)|\le O(\varepsilon\delta\mu) \tag{8}$$

for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_2)\times V^-$,

where $V^-=\{\delta\mu\in V:\delta t_0\le 0\}$ and $\lim\limits_{\varepsilon\to 0}O(\varepsilon\delta\mu)/\varepsilon=g(\delta\mu)$ uniformly for $\delta\mu\in V^-$ and $|g(\delta\mu)|$ is bounded on V^-.

Moreover,

$$\Delta x(t_{00})=\varepsilon\{\delta\varphi(t_{00})+[\dot{\varphi}_0(t_{00})-f[t_{00}]]\delta t_0\}+o(\varepsilon\delta\mu). \tag{9}$$

Lemma 2. There exist numbers $\varepsilon \in (0,\varepsilon_1)$ and $\delta_2 \in (0,\delta_1)$ such that the inequality (8) is valid for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_2)\times V^+$, where $V^+ = \{\delta\mu \in V : \delta t_0 \ge 0\}$. Moreover,
$\Delta x(t_0) = \varepsilon\{\delta\varphi(t_{00}) + [\dot\varphi_0(t_{00}) - f[t_{00}]]\delta t_0\} + o(\varepsilon\delta\mu)$.

Lemmas 1 and 2 can be proved in analogy to Lemma 2.4 (Tadumadze & Alkhazishvili, 2004).

4. Proof of Theorem 1

Let $\delta t_0 \le 0$ then the function $\Delta x(t)$ satisfies the equation

$$\begin{aligned}\dot{\Delta x}(t) &= f[t,x_0+\Delta x] - f[t] = f_x[t]\Delta x(t) + f_{x_1}[t]\Delta x(t-\tau_0) + \\ &+ \varepsilon\big[f_u[t]\delta u(t) + f_{u_1}[t]\delta u(t-\theta_0) + \\ &+ f_{u_1}[t]\dot u_0(t-\theta_0)\delta\theta\big] + r(t;\varepsilon\delta\mu)\end{aligned} \tag{10}$$

on the interval $[t_{00}, t_{10}+\delta_2]$, where

$$\begin{aligned}r(t;\varepsilon\delta\mu) &= f[t,x_0+\Delta x] - f[t] - f_x[t]\Delta x(t) - f_{x_1}[t]\Delta x(t-\tau_0) - \\ &- \varepsilon\big[f_u[t]\delta u(t) + f_{u_1}[t]\delta u(t-\theta_0) + \\ &+ f_{u_1}[t]\dot u_0(t-\theta_0)\delta\theta\big],\end{aligned} \tag{11}$$

$f[t,x_0+\Delta x] = f(t, x_0(t)+\Delta x(t), x_0(t-\tau)+\Delta x(t-\tau), u(t), u(t-\theta))$, $\tau = \tau_0 + \varepsilon\delta\tau$,
$u(t) = u_0(t) + \varepsilon\delta u(t)$, $\theta = \theta_0 + \varepsilon\delta\theta$.

By using the Cauchy formula, one can represent the solution of equation (10) in the form

$$\begin{aligned}\Delta x(t) &= Y(t_{00};t)\Delta x(t_{00}) + \\ &+ \varepsilon\int_{t_{00}}^{t} Y(\xi;t)\big[f_u[\xi]\delta u(\xi) + f_{u_1}[\xi]\delta u(\xi-\theta_0) + f_{u_1}[\xi]\dot u_0(\xi-\theta_0)\delta\theta\big]d\xi + \sum_{p=0}^{1} R_p\end{aligned} \tag{12}$$

where

$$R_0 = R_0(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t) f_{x_1}[\xi+\tau_0]\Delta x(\xi)d\xi\ , \tag{13}$$

$$R_1 = R_1(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi;t) r(\xi;\varepsilon\delta\mu)d\xi\ ; \tag{14}$$

and $Y(\xi;t)$ is the matrix function satisfying equation(6) and condition (7). The function $Y(\xi;t)$ is continuous on the set $\{(\xi,t): t_{00} \le \xi \le t, t \in [t_{00}, t_{10}+\delta_2]\}$ by Lemma 2.6 in (Tadumadze, 2017, p.32). Therefore,

$$Y(t_{00};t)\Delta x(t_{00}) = \varepsilon Y(t_{00};t)\{\delta\varphi(t_{00}) + [\dot\varphi_0(t_{00}) - f[t_{00}]]\delta t_0\} + o(t;\varepsilon\delta\mu) \tag{15}$$

(see (9)), where $o(t;\varepsilon\delta\mu) = Y(t_{00};t)o(\varepsilon\delta\mu)$.

One can readily see that

$$\begin{aligned}R_0 &= \varepsilon\int_{t_{00}-\tau_0}^{t_0} Y(\xi+\tau_0;t) f_{x_1}[\xi+\tau_0]\delta\varphi(\xi)d\xi + \int_{t_0}^{t_{00}} Y(\xi+\tau_0;t) f_{x_1}[\xi+\tau_0]\Delta x(\xi)d\xi = \\ &= \varepsilon\int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t) f_{x_1}[\xi+\tau_0]\delta\varphi(\xi)d\xi + o(t;\varepsilon\delta\mu).\end{aligned} \tag{16}$$

(see (13), (3),(8)).
We introduce the notations:

$$f[t;s,\varepsilon\delta\mu]=f(t,x_0(t)+s\Delta x(t),x_0(t-\tau_0)+s(x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau)),$$
$$u_0(t)+\varepsilon s\delta u(t),u_0(t-\theta_0)+s(u_0(t-\theta)-u_0(t-\theta_0)+\varepsilon\delta u(t-\theta)));$$
$$\sigma(t;s,\varepsilon\delta\mu)=f_x[t;s,\varepsilon\delta\mu]-f_x[t],\ \sigma_1(t;s,\varepsilon\delta\mu)=f_{x_1}[t;s,\varepsilon\delta\mu]-f_x[t],$$
$$\vartheta(t;s,\varepsilon\delta\mu)=f_u[t;s,\varepsilon\delta\mu]-f_u[t],\ \vartheta_1(t;s,\varepsilon\delta\mu)=f_{u_1}[t;s,\varepsilon\delta\mu]-f_{u_1}[t].$$

It is easy to see that

$$f[t,x_0+\Delta x]-f[t]=\int_0^1\frac{d}{ds}f[t;s,\varepsilon\delta\mu]ds=\int_0^1\{f_x[t;s,\varepsilon\delta\mu]\Delta x(t)+f_{x_1}[t;s,\varepsilon\delta\mu]\big(x_0(t-\tau)-$$
$$-x_0(t-\tau_0)+\Delta x(t-\tau)\big)+\varepsilon f_u[t;s,\varepsilon\delta\mu]\delta u(t)+f_{u_1}[t;s,\varepsilon\delta\mu]\big(u_0(t-\theta)-u_0(t-\theta_0)+$$
$$+\varepsilon\delta u(t-\theta)\big)\}ds=\left[\int_0^1\sigma(t;s,\varepsilon\delta\mu)ds\right]\Delta x(t)+\left[\int_0^1\sigma_1(t;s,\varepsilon\delta\mu)ds\right]\big(x_0(t-\tau)-x_0(t-\tau_0)+$$
$$+\Delta x(t-\tau)\big)+\varepsilon\left[\int_0^1\vartheta(t;s,\varepsilon\delta\mu)ds\right]\delta u(t)+$$
$$\left[\int_0^1\vartheta_1(t;s,\varepsilon\delta\mu)ds\right]\big(u_0(t-\theta)-u_0(t-\theta_0)+\varepsilon\delta u(t-\theta)\big)+$$
$$+f_x[t]\Delta x(t)+f_{x_1}[t]\big(x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau)\big)+\varepsilon f_u[t]\delta u(t)+$$
$$+f_{u_1}[t]\big(u_0(t-\theta)-u_0(t-\theta_0)+\varepsilon\delta u(t-\theta)\big).$$

By taking account of last relation for $t\in[t_{00},t_{10}+\delta_2]$ we have

$$R_1=\int_{t_{00}}^{t}Y(\xi;t)\Big\{\left[\int_0^1\sigma(\xi;s,\varepsilon\delta\mu)ds\right]\Delta x(\xi)+\left[\int_0^1\sigma_1(\xi;s,\varepsilon\delta\mu)ds\right]\big(x_0(\xi-\tau)-x_0(\xi-\tau_0)+$$
$$+\Delta x(\xi-\tau)+\varepsilon\left[\int_0^1\vartheta(\xi;s,\varepsilon\delta\mu)ds\right]\delta u(\xi)+\left[\int_0^1\vartheta_1(\xi;s,\varepsilon\delta\mu)ds\right]\big(u_0(\xi-\theta)-u_0(\xi-\theta_0)+$$
$$+\varepsilon\delta u(\xi-\theta)\big)+f_x[\xi]\Delta x(\xi)+f_{x_1}[\xi]\big(x_0(\xi-\tau)-x_0(\xi-\tau_0)+\Delta x(\xi-\tau)\big)+\varepsilon f_u[\xi]\delta u(\xi)+$$
$$+f_{u_1}[\xi]\big(u_0(\xi-\theta)-u_0(\xi-\theta_0)+\varepsilon\delta u(\xi-\theta)\big)-f_x[\xi]\Delta x(\xi)-f_{x_1}[\xi]\Delta x(\xi-\tau_0)-\varepsilon f_u[\xi]\delta u(\xi)-$$
$$-\varepsilon f_{u_1}[\xi]\big(\delta u(\xi-\theta_0)+\dot u_0(\xi-\theta_0)\delta\theta\big)\Big\}d\xi=\sum_{p=2}^{8}R_p,$$

where

$$R_2=\int_{t_{00}}^{t}Y(\xi;t)\hat\sigma(\xi;\varepsilon\delta\mu)\Delta x(\xi)d\xi,\hat\sigma=\sigma(\xi;\varepsilon\delta\mu)=\int_0^1\sigma(\xi;s,\varepsilon\delta\mu)ds;$$
$$R_3=\int_{t_{00}}^{t}Y(\xi;t)\hat\sigma_1(\xi;\varepsilon\delta\mu)\big(x_0(\xi-\tau)-x_0(\xi-\tau_0)+\Delta x(\xi-\tau)\big)d\xi,$$
$$\hat\sigma_1=\int_0^1\sigma_1(\xi;s,\varepsilon\delta\mu)ds;\ R_4=\varepsilon\int_0^1Y(\xi;t)\hat\vartheta(\xi;\varepsilon\delta\mu)\delta u(\xi),\hat\vartheta=\int_0^1\vartheta(\xi;s,\varepsilon\delta\mu)d\xi;$$

$$R_5 = \int_{t_{00}}^{t} Y(\xi;t) f_{x_1}[\xi]\big(x_0(\xi-\tau) - x_0(\xi-\tau_0)\big)d\xi; R_6 = \int_{t_{00}}^{t} Y(\xi;t) f_{x_1}[\xi]\big(\Delta x(\xi-\tau) - \Delta x(\xi-\tau_0)\big)d\xi;$$

$$R_7 = \int_{t_{00}}^{t} Y(\xi;t)\hat{\vartheta}_1(\xi;t)\big(u_0(\xi-\theta) - u_0(\xi-\theta_0) + \varepsilon\delta u(\xi-\theta)\big)d\xi, \hat{\vartheta}_1 = \int_0^1 \vartheta_1(\xi;s,\varepsilon\delta\mu)ds;$$

$$R_8 = \int_{t_{00}}^{t} Y(\xi;t) f_{u_1}[\xi]\big(u_0(\xi-\theta) - u_0(\xi-\theta_0) - \varepsilon\dot{u}_0(\xi-\theta_0)\delta\theta + \varepsilon(\delta u(\xi-\theta) - \delta u(\xi-\theta_0))\big)d\xi$$

(see (14)). The function $x_0(t)$ is continuously differentiable on $[\hat{\tau}, t_{00}) \cup (t_{00}, t_{10} + \delta_2]$, therefore for each fixed $\xi \in [t_{00}, t_{00} + \tau_0) \cup (t_{00} + \tau_0, t_{10} + \delta_2]$ we get

$$x_0(\xi-\tau) - x_0(\xi-\tau_0) = \int_{\xi}^{\xi-\varepsilon\delta\tau} \dot{x}_0(\varsigma-\tau_0)d\varsigma = -\varepsilon\dot{x}_0(\xi-\tau_0)\delta\tau + \gamma(\xi;\varepsilon\delta\mu), \tag{17}$$

where

$$\lim_{\varepsilon\to 0} \frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon} = 0 \text{ uniformly for } \delta\mu \in V^-. \tag{18}$$

From (17) it follows

$$|x_0(\xi-\tau) - x_0(\xi-\tau_0)| \le O(\varepsilon\delta\mu) \text{ and } \left|\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}\right| \le C = const, \tag{19}$$

$\forall(t,\varepsilon,\delta\mu) \in [t_{00}, t_{00} + \tau_0) \cup (t_{00} + \tau_0, t_{10} + \delta_2] \times (0,\varepsilon_1) \times V^-$.

It is clear that

$$|\Delta x(\xi-\tau) - \Delta x(\xi-\tau_0)| = \begin{cases} o(\xi;\varepsilon\delta\mu) & for\ \xi \in [t_{00}, \rho_1], \\ O(\xi;\varepsilon\delta\mu) & for\ \xi \in [\rho_1, \rho_2], \end{cases} \tag{20}$$

where $\rho_1 = \min\{t_0 + \tau, t_{00} + \tau_0\}$ and $\rho_2 = \max\{t_{00} + \tau, t_{00} + \tau_0\}$.

Let $\xi \in [\rho_2, t_{10} + \delta_2]$ then $\xi - \tau \ge t_{00}$ and $\xi - \tau_0 \ge t_{00}$, therefore

$$|\Delta x(\xi-\tau) - \Delta x(\xi-\tau_0)| \le \left|\int_{\xi-\tau_0}^{\xi-\tau} \left|\dot{\Delta}x(t)\right|dt\right| = \left|\int_{\xi-\tau_0}^{\xi-\tau} |f(t,x_0+\Delta x) - f[t]|dt\right| \le$$

$$\le \varepsilon|\delta\tau|\sup\{|f(t,x_0+\Delta x) - f[t]| : t \in [t_{00}, t_{10} + \delta_2]\} = o(\xi;\varepsilon\delta\mu). \tag{21}$$

According to (17)-(21) for the expressions

$$R_2 = R_3 = R_4 = R_6 = R_7 = R_8 = o(t;\varepsilon\delta\mu)$$

and

$$R_5 = -\varepsilon\left\{\left[\int_{t_{00}}^{t} Y(\xi;t) f_{x_1}[\xi]\dot{x}_0(\xi-\tau_0)d\xi\right]\delta\tau + \int_{t_{00}}^{t} Y(\xi;t) f_{x_1}[\xi]\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}d\xi\right\} =$$

$$= -\varepsilon\left[\int_{t_{00}}^{t} Y(\xi;t) f_{x_1}[\xi]\dot{x}_0(\xi-\tau_0)d\xi\right]\delta\tau + o(t;\varepsilon\delta\mu).$$

Thus,

$$R_1 = -\varepsilon\left[\int_{t_{00}}^{t} Y(\xi;t) f_{x_1}[\xi]\dot{x}_0(\xi-\tau_0)d\xi\right]\delta\tau + o(t;\varepsilon\delta\mu). \tag{22}$$

From (12) by virtue of (13) and (22) we conclude that for a fixed $\hat{t}_0 \in (t_{00}, t_{10})$ and for each $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times V^-$ the formula (4) is valid on the interval $[\hat{t}_0, t_{10} + \delta_2]$, where $\delta x(t; \delta\mu)$ has the form (5).

Let $\delta t_0 \geq 0$ and let $\hat{t}_0 \in (t_{00}, t_{10})$ be a fixed point, and let $\varepsilon \in (0, \varepsilon_1)$ be so small that $t_0 < \hat{t}_0$ for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times V^+$. The function $\Delta x(t)$ satisfies equation (10) on the interval $[t_0, t_{10} + \delta_2]$; therefore we can represent it in the form

$$\Delta x(t) = Y(t_0; t)\Delta x(t_0) + \varepsilon \int_{t_0}^{t} Y(\xi; t)\left[f_u[\xi]\delta u(\xi) + f_{u_1}[\xi]\delta u(\xi - \theta_0) + f_{u_1}[\xi]\dot{u}_0(\xi - \theta_0)\delta\theta \right] d\xi + \sum_{p=0}^{1} R_p(t; t_0, \varepsilon\delta\mu). \tag{23}$$

The matrix function $Y(\xi; t)$ is continuous on $(t_{00}, \hat{t}_0) \times [\hat{t}_0, t_{10} + \delta_2]$; therefore on the interval $[\hat{t}_0, t_{10} + \delta_2]$ is valid

$$Y(t_0; t)\Delta x(t_0) = \varepsilon Y(t_{00}; t)\left\{ \delta\varphi(t_{00}) + \left[\dot{\varphi}_0(t_{00}) - f[t_{00}] \right]\delta t_0 \right\} + o(t; \varepsilon\delta\mu)$$

(see Lemma 2).

In a similar way, with nonessential changes, for $t \in [\hat{t}_0, t_{10} + \delta_2]$ one can prove

$$R_0(t; t_0, \varepsilon\delta\mu) = -\varepsilon \left[\int_{t_{00}}^{t} Y(\xi; t) f_{x_1}[\xi]\dot{x}_0(\xi - \tau_0) d\xi \right] \delta\tau + o(t; \varepsilon\delta\mu)$$

and

$$R_1(t; t_0, \varepsilon\delta\mu) = -\varepsilon \left[\int_{t_{00}}^{t} Y(\xi; t) f_{x_1}[\xi]\dot{x}_0(\xi - \tau_0) d\xi \right] \delta\tau + o(t; \varepsilon\delta\mu).$$

Finally, we note that

$$\varepsilon \int_{t_0}^{t} Y(\xi; t)\left[f_u[\xi]\delta u(\xi) + f_{u_1}[\xi]\delta u(\xi - \theta_0) + f_{u_1}[\xi]\dot{u}_0(\xi - \theta_0)\delta\theta \right] d\xi =$$

$$= \varepsilon \int_{t_{00}}^{t} Y(\xi; t)\left[f_u[\xi]\delta u(\xi) + f_{u_1}[\xi]\delta u(\xi - \theta_0) + f_{u_1}[\xi]\dot{u}_0(\xi - \theta_0)\delta\theta \right] d\xi + o(t; \varepsilon\delta\mu).$$

From (23) on the basis of last relations we conclude that for each $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times V^+$ formula(4) is valid on $[\hat{t}_0, t_{10} + \delta_2]$, where $\delta x(t; \delta\mu)$ has the form (5). Theorem 1 is proved.

5. Conclusion

The formulas (4) and (5) can be used under investigation of delay optimization problems, namely in proving the necessary conditions for optimality of delays τ and θ. Moreover, they allow one to find analytical relations between of solutions of the initial and perturbed equations (sensitivity analysis of models) and to construct an approximate solution of the perturbed equation.

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